Recall

Define $\Phi(s) = s\ln(s)$. Then the entropy w.r.t. measure μ is

$$\mathrm{Ent}_\mu(f) := \int \Phi \circ f \, d\mu - \Phi\left(\int f d\mu\right).$$

Define $D_\nu(F, G) = \mathbb{E}_\nu[\nabla F \cdot \nabla G]$, and $D_\nu(F) = D_\nu(F, F)$.
ν satisfies a log-Sob inequality (LSI) if $\exists \gamma > 0$ s.t. $\forall F \in C_c^\infty(X; \mathbb{R}_+)$,

$$\mathrm{Ent}_\nu(F) \leq \frac{2}{\gamma} D_\nu(\sqrt{F}).$$

(Last time: $\mathrm{Ent}_\nu(f^2) \leq 2C \int_X |\nabla f|^2 d\nu$).

Glauber-Langevin Dynamics

Notation: $Y^X = \{$ all functions $f: X \longrightarrow Y \}$.

Ex. $\mathbb{R}^{\mathbb{N}} = \{ f : \mathbb{N} \longrightarrow \mathbb{R} \} = \{$ all real-valued sequences $\}$.

we used this notation differently last week!

Def: For a finite (large) set $\Lambda \subseteq \mathbb{Z}^d$, $\varphi \in \mathbb{R}^\Lambda$ is called a cts. spin field.

Notation: Since $|\Lambda| =: N < \infty$, $\mathbb{R}^\Lambda \approx \mathbb{R}^N$. For $f : \mathbb{R}^\Lambda \longrightarrow \mathbb{R}$ nice enough, we define

$$\int_{\mathbb{R}^\Lambda} f(\varphi)\, d\varphi = \int_{\mathbb{R}^N} f(\lambda_1 \longmapsto x_1, \ldots, \lambda_N \longmapsto x_N)\, dx_1 \ldots dx_N.$$

Moral: Lebesgue measure on $\mathbb{R}^\Lambda$!

Def: $\nu \in P(\mathbb{R}^{\Lambda})$ is an equilibrium Gibbs measure if there exists $H : \mathbb{R}^{\Lambda} \longrightarrow \mathbb{R}$ s.t.

$$\nu(A) \propto \int_A e^{-H(\varphi)} d\varphi.$$

H is called the action or Hamiltonian.

Recall: Let X be some infinite-dimensional state space, and let G be an operator on $\mathbb{R}^X$. G is called a generator.

1. We define the semigroup T^G by

$$T^G = \{ T_t = e^{tG} = \sum_{i=0}^{\infty} \frac{t^i G^i}{i!} : t \geq 0 \}.$$

2. Let Y_t equal the RV that encodes our state at time t. If Y_0 has law m_0, then Y_t has law $m_t := T_t m_0$.

3. We say ν is an invariant measure w.r.t. this Markov chain if the following holds: If $m_t = \nu$, then $\forall s > t$, $m_s = \nu$.

Fact: Consider the state space $X := \mathbb{R}^{\Lambda}$. Then the solution to the Glauber-Langevin dynamics

$$d\varphi_t = -\nabla H(\varphi_t)\, dt + \sqrt{2}\, dB_t$$

is a Markov process with generator

$$\Delta^H := \sum_{x \in \Lambda} \frac{\partial^2}{\partial \varphi_x^2} - \frac{\partial H}{\partial \varphi_x} \frac{\partial}{\partial \varphi_x}.$$

Fact: The dynamics has invariant measure $\nu(A) \propto \int_A e^{-H(\varphi)} d\varphi$. Furthermore, $m_t \longrightarrow \nu$.

What do we mean by $m_t \longrightarrow \nu$?

Suppose ν satisfies an LSI with constant γ.

Define $F_t = \frac{\partial m_t}{\partial \nu}$, and note $\mathbb{H}(m_t | \nu) = \mathrm{Ent}_\nu(F_t)$.

We will show

$$\mathbb{H}(m_t | \nu) \le e^{-2\lambda t} \mathbb{H}(m_0 | \nu).$$

To do this, we use the following result.

Prop (De Bruijn identity): Let $F : X \to \mathbb{R}$, and define

$$F_t(\varphi) = \mathbb{E}_{\varphi_0 = \varphi}[F(\varphi_t)].$$

Then $\frac{d}{dt} \mathrm{Ent}_\nu(F_t) = -D_\nu(\ln(F_t), F_t) \le 0$.

Proof: * Markov chain theory * tells us F_t satisfies the backward Kolmogorov equation, i.e. $\frac{d}{dt} F_t = \Delta^H F_t$.
As a result,

$$\frac{d}{dt} \mathbb{E}_\nu[F_t] = \mathbb{E}_\nu[\Delta^H F_t] \overset{IBP}{=} -\mathbb{E}_\nu[\nabla F_t \cdot \nabla 1] = 0. \qquad (1)$$

This implies

$$\frac{d}{dt} \mathrm{Ent}_\nu(F_t) = \frac{d}{dt}\Big[\mathbb{E}_\nu[\Phi \circ F_t] - \cancelto{0}{\Phi(\mathbb{E}_\nu[F_t])}\Big]$$

$$= \mathbb{E}_\nu\Big[\Phi'(F_t) \frac{d}{dt} F_t\Big]$$

$$= \mathbb{E}_\nu\Big[\Phi'(F_t) \Delta^H F_t\Big]$$

$$\overset{IBP}{=} -\mathbb{E}_\nu\Big[\nabla \Phi'(F_t) \cdot \nabla \Delta^H F_t\Big]$$

$$= -D_\nu(\Phi'(F_t), F_t)$$

$$= -D_\nu(\ln(F_t)+1, F_t)$$

$$= -D_\nu(\ln(F_t), F_t).$$

Finally, by calculus,

$$\star \quad D_\nu(\ln(F_t), F_t) = \mathbb{E}_\nu\left[\nabla \ln(F_t) \cdot \nabla F_t\right]$$

$$= \mathbb{E}_\nu\left[\frac{\nabla F_t \cdot \nabla F_t}{F_t}\right]$$

$$= 4\,\mathbb{E}_\nu\left[|\nabla \sqrt{F_t}|^2\right].$$

$$\geq 0.$$

$\square$

Suppose ν satisfies an LSI with constant γ. Then

$$\frac{d}{dt}\operatorname{Ent}_\nu(F_t) \leq \frac{2}{\gamma} D_\nu(\sqrt{F_t})$$

$$= \frac{2}{\gamma}\mathbb{E}_\nu\left[\nabla\sqrt{F_t} \cdot \nabla\sqrt{F_t}\right]$$

$$\overset{\star}{=} \frac{1}{2\gamma} D_\nu(\ln(F_t), F_t)$$

$$= \frac{1}{2\gamma}\operatorname{Ent}_\nu(F_t).$$

So, by Gronwall's, $\operatorname{Ent}_\nu(F_t) \leq e^{-2\gamma t}\operatorname{Ent}_\nu(F_0)$.

Thm (Bakry - Émery): Let $\nu \in P(X)$ be an equilibrium Gibbs measure with Hamiltonian H. Further suppose $\exists \lambda > 0$ s.t. $\forall \varphi \in X$,

$$\operatorname{Hess} H(\varphi) \geq \lambda \, \mathrm{id}.$$

Then ν satisfies an LSI with constant λ.

Proof: Fix $F \in C_c^\infty(X; \mathbb{R}_+)$, and define $F_t(\varphi) = \mathbb{E}_{\varphi = \varphi_0}[F(\varphi_t)]$.
First, we prove

$$D_\nu(\sqrt{F_t}) \leq e^{-2\lambda t} D_\nu(\sqrt{F}). \qquad ②$$

Calculations imply $\frac{d}{dt} D_\nu(\sqrt{F_t}) \leq -2\lambda D_\nu(\sqrt{F})$, which by Gronwell's, gives us ②.
As a result,

more skippable calculations here

$$\begin{aligned}
\operatorname{Ent}_\nu(F) &= \mathbb{E}_\nu\left[\Phi(F_0) - \Phi(F_\infty)\right] \\
&= \mathbb{E}_\nu\left[-\int_0^\infty \frac{d}{dt}\Phi(F_t)\,dt\right] \\
&= -\int_0^\infty \frac{d}{dt}\mathbb{E}_\nu[\Phi(F_t)]\,dt \\
&= \int_0^\infty 4\,\mathbb{E}_\nu\left[|\nabla \sqrt{F_t}|^2\right] dt \\
&= \int_0^\infty 4 D_\nu(\sqrt{F_t})\,dt \\
&\leq 4 D_\nu(\sqrt{F}) \int_0^\infty e^{-2\lambda t}\,dt \\
&= \frac{2}{\lambda} D_\nu(\sqrt{F}).
\end{aligned}$$

$\square$