Concentration of Measure

Setup: (X,d), $\mu \in P(X)$.
For $A \subseteq X$, $r > 0$,

$$A_r := \{ x : \inf_{a \in A} d(x,a) < r \}.$$

Def: We define the concentration function as

$$\alpha(r) := \sup\{ 1 - \mu(A_r) : A \subseteq X, \mu(A) \geq \tfrac{1}{2} \}$$

$$= 1 - \inf\{ \mu(A_r) : A \subseteq X, \mu(A) \geq \tfrac{1}{2} \}$$

find set A with measure greater than 1/2
s.t. measure of "fuzzy ball" is minimized.

Ex. $(X,d) = ([0,1], |\cdot|)$, $\mu = \text{Unif}$.

For $A \subseteq X$, $\mu(A) \geq \frac{1}{2}$, the easiest way to minimize $\mu(A_r)$ is to make A an interval.
Set $A = [0, \frac{1}{2}]$, then $A_r = [0, \frac{1}{2} + r)$, $\mu(A_r) = \frac{1}{2} + r$,

$$\alpha(r) = \tfrac{1}{2} - r.$$

Ex. $(X,d) = (\mathbb{R}, |\cdot|)$, $\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_i$, n even.

$\mu(A_r)$ is minimized when A is an interval.
Set $A = [0, \frac{n}{2}] \Rightarrow \mu(A) = \frac{1}{2}$.
If $r \leq 1$, A_r does not contain any "new" integers, so $\mu(A_r) = \frac{1}{2}$.
If $r > 1$, $\mu(A_r)$ contains $\lfloor r \rfloor$ "new" integers, so $\mu(A_r) = \frac{1}{2} + \frac{1}{n} \lfloor r \rfloor$.
Thus,

$$\alpha(r) = \begin{cases} \frac{1}{2} & r \leq 1 \\ \frac{1}{2} - \frac{1}{n} \lfloor r \rfloor & r > 1. \end{cases}$$

Intuition: For a fixed r, $\alpha(r)$ measures "how much" of X is left when you expand sets w/ measure $\frac{1}{2}$ by r.
If a measure is more concentrated, we can expand sets without taking away much mass, so α will be larger.

Fact: $\lim_{r\to\infty} \alpha(r) = 0$.

Def: Let $F: X \to \mathbb{R}$ measurable. $m_F \in \mathbb{R}$ is a median of F if

$$\mu(\{F \le m_F\}) \ge \tfrac{1}{2} \quad \text{and} \quad \mu(\{F \ge m_F\}) \ge \tfrac{1}{2}.$$

Def: Let $F: X \to \mathbb{R}$ cts. The modulus of continuity is $\omega_F : (0,\infty) \longrightarrow \mathbb{R}$ defined by

$$\omega_F(\eta) = \sup\{\,|F(x) - F(y)| : d(x,y) < \eta\,\}.$$

Deviation Inequalities

Inequality 1: For $\eta > 0$, $\mu(\{F > m_F + \omega_F(\eta)\}) \le \alpha(\eta)$.

Proof: Define $A = \{F \le m_F\}$ and fix $a \in A$.
Choose $x \in A_\eta$.
Then

$$F(x) \le F(a) + \omega_F(\eta) \le m_F + \omega_F(\eta),$$

i.e. $x \in \{F \le m_F + \omega_F(\eta)\}$.
So $A_\eta \subseteq \{F \le m_F + \omega_F(\eta)\}$

Since $\mu(A) \geq \frac{1}{2}$,

$$\alpha(\eta) \geq 1 - \mu(A_\eta)$$

$$\geq 1 - \mu(\{F \leq m_F + \omega_F(\eta)\})$$

$$= \mu(\{F > m_F + \omega_F(\eta)\}$$

$\square$

Inequality 2: If F is Lipschitz, $\forall r > 0$,

$$\mu(\{F \geq m_F + r\}) \leq \alpha\left(\frac{r}{\|F\|_{Lip}}\right).$$

and $\mu(\{|F - m_F| \geq r\}) \leq 2\alpha\left(\frac{r}{\|F\|_{Lip}}\right)$.

Proof: Again, define $A = \{F \leq m_F\}$ and fix $a \in A$.
Choose $x \in A_r$.
Then

$$F(x) \leq F(a) + \|F\|_{Lip}\, d(x,a) \leq m_F + \|F\|_{Lip}\, r,$$

i.e. $x \in \{F \leq m_F + \|F\|_{Lip}\, r\}$.
So $A_r \subseteq \{F \leq m_F + \|F\|_{Lip}\, r\}$.
Since $\mu(A) \geq \frac{1}{2}$,

$$\alpha(r) \geq 1 - \mu(A_r)$$

$$\geq 1 - \mu(\{F \leq m_F + \|F\|_{Lip}\, r\})$$

$$= \mu(\{F > m_F + \|F\|_{Lip}\, r\}).$$

Setting $r' = \frac{r}{\|F\|_{Lip}}$, we obtain the result.
Similarly, we obtain

$$\mu(\{F \leq m_F - r\}) \leq \alpha\left(\frac{r}{\|F\|_{Lip}}\right)$$

In sum, $\mu(\{|F - m_F| \leq r\}) \leq 2\alpha\left(\frac{r}{\|F\|_{Lip}}\right)$.

$\square$

Cor 1.4: Let $A, B \subseteq X$ nonempty, Borel. Then

$$\mu(A)\mu(B) \leq 4\alpha\left(\frac{d(A,B)}{2}\right).$$

Proof: If $A \cap B \neq \emptyset$, then $4\alpha\left(\frac{d(A,B)}{2}\right) = 4\alpha(0) = 2$, so the inequality holds.
Define $r = \frac{d(A,B)}{2} > 0$, $F(x) = d(x, B)$.

If $x \in A, y \in B$, then $|F(x) - F(y)| = F(x) \geq 2r$.
Thus,

$$\mu(A)\mu(B) \leq (\mu \otimes \mu)(\{(x,y) : |F(x) - F(y)| \geq 2r\})$$

$$\leq 2\mu(\{|F - m_F| \geq r\})$$

inequality 2

$$\leq 4\alpha(r)$$

$$= 4\alpha\left(\frac{d(A,B)}{2}\right).$$

$\square$

Prop 1.7: Suppose $\exists \beta : (0,\infty) \to [0,\infty)$ s.t. $\forall F : X \to \mathbb{R}$ bounded and 1-Lipschitz, $\forall r > 0$,

$$\mu(\{F \ge \int F \, d\mu + r\}) \le \beta(r).$$

Then, $\forall$ Borel sets A with $\mu(A) > 0$ and $\forall r > 0$,

$$1 - \mu(A_r) \le \beta(\mu(A) r).$$

In particular, $\alpha(r) \le \beta(\frac{r}{2})$.

Proof: Choose Borel set A with $\mu(A) > 0$ and fix $r > 0$.
Define $F(x) = \min(d(x, A), r)$, and note F is bounded and 1-Lipschitz.
Furthermore,

$$\int F \, d\mu = \int_{A_r} d(x, A) \, d\mu + \int_{A_r^c} r \, d\mu$$

$$< r(\cancel{\mu(A_r)} - \mu(A)) + r(1 - \cancel{\mu(A_r)})$$

$$= r(1 - \mu(A)).$$

Thus, $\int F \, d\mu + r\mu(A) < r$.
This gives

$$\begin{aligned} 1 - \mu(A_r) &= \mu(A_r^c) \\ &= \mu(\{F = r\}) \\ &\le \mu(\{F > \int F \, d\mu + r\mu(A)\}) \\ &\le \beta(\mu(A) r). \end{aligned}$$

Finally, if $\mu(A) \ge \frac{1}{2}$, $1 - \mu(A_r) \le \beta(\frac{r}{2})$, so $\alpha(r) \le \beta(\frac{r}{2})$.

$\square$

Expansion Coefficients

Def: The expansion coefficient of order $\varepsilon > 0$ is

$$\mathrm{Exp}_\mu(\varepsilon) = \inf\{\eta \geq 1 : \forall A \subseteq X \text{ s.t. } \mu(A_\varepsilon) \leq \tfrac{1}{2},\ \mu(A_\varepsilon) \geq \eta\mu(A)\}.$$

Ex. Fix $k \in \mathbb{N}$, and choose $A \subseteq X$ s.t. $\mu(A_{k\varepsilon}) \leq \frac{1}{2}$.
Then,

$$\begin{aligned}(\mathrm{Exp}_\mu(\varepsilon))^k \mu(B) &\leq (\mathrm{Exp}_\mu(\varepsilon))^{k-1}\mu(B_\varepsilon) \\ &\leq (\mathrm{Exp}_\mu(\varepsilon))^{k-2}\mu(B_{2\varepsilon}) \\ &\ \vdots \\ &\leq \mu(B_{k\varepsilon}).\end{aligned}$$

$(B_\varepsilon)_\varepsilon \subseteq B_{2\varepsilon}$.

So $\mu(B) \leq (\mathrm{Exp}_\mu(\varepsilon))^{-k}\mu(B_{k\varepsilon}) \leq \frac{1}{2}(\mathrm{Exp}_\mu(\varepsilon))^{-k}$.

Prop 1.13: If $\exists\, \varepsilon > 0$ s.t. $\mathrm{Exp}_\mu(\varepsilon) \geq \eta > 1$, then

$$\alpha(r) \leq \frac{\eta}{2} e^{-r(\ln(\eta))/\varepsilon}.$$